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A new method for designing a developable surface utilizing the surface pencil through a given curve

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Abstract

This paper proposes a new method for designing a developable surface by constructing a surface pencil passing through a given curve, which is quite in accord with the practice in industry design and manufacture. By utilizing the Frenet trihedron frame, we derive the necessary and sufficient conditions to construct a developable surface through a given curve. Considering the requirements in shoemaking and garment-manufacture industries, we also study the special case of specifying the given curve as a geodesic. The given geodesic can be classified into three types corresponding to each type of developable surface. We also present the polynomial representation of the developable surface. The algorithm is convenient and efficient for applications in engineering.

Keywords: CAD/CAM; Developable surface; Geodesic; Frenet frame

1. Introduction

Developable surfaces, which can be developed onto a plane without stretching and tearing, have natural applications in many areas of engineering and manufacturing, including modeling of ship hulls [1], apparel [2], automobile components [3], and so on. In computer graphics, many objects can be approximated by piecewise continuous developable surfaces.

Many studies on designing with developable surfaces have been reported. Given a set of data points, Redont [4] built a family of circular cones to approximate them, and then formed the desired developable surface using patches of the circular cones. Bodduluri and Ravani [5], Pottmann and Farin [6], Pottmann and Wallner [7] constructed developable surfaces in terms of plane geometry using the concept of duality between points and planes in 3D projective space. They provided a compact representation for developable surfaces in the dual form. Chu and Séquin [8], Aumann [9] developed a new approach to geometric design of developable Bézier patches based on the de Casteljau algorithm. Their algorithms are suitable for constructing developable surfaces with boundary curves of polynomial form.

However, most existing work on developable surfaces concentrates on providing designing methods but neglecting the practical requirement in industry that a developable surface passing through a given spatial curve is needed and the curve is at the same time a geodesic of the surface. For example, in shoemaking industry, the girth is used to measure the shoe size. And in garment-manufacture industry, waist line is used. The girth and the waist line can be considered to be a geodesic on the shoe surface and the cloth, respectively. Wang et al. [10] provided a method for designing a surface pencil with the given curve as a common geodesic, among which each surface can be a candidate for fashion designing. However, the developability of surfaces was not considered, which is important in real applications.

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For example, shoes' surface or any part generally is developable or approximatively developable, that is, it can be flattened to a plane with minimum or even no distortion, because the material, i.e. the leather, is a plane. Similarly, the material of dresses' surface, i.e. the cloth, is a plane. Therefore, research on designing a developable surface from a given curve is attractive. Inspired by Wang et al. [10], in this paper, we propose a new method for designing developable surfaces from a given curve. The problem of requiring the curve to be a geodesic is also studied. And the designing of three types of developable surfaces: cylinders, cones and tangent surfaces is introduced.

2. Fundamentals

Developable surfaces can be briefly introduced as special cases of ruled surfaces. The ruled surface is a kind of surface commonly used in CAD/CAM systems. A ruled surface P carries a one parameter family of straight lines L. These lines are called generators or generating lines. The general parameterization of a ruled surface P can be expressed as

$$P: \boldsymbol{P}(r,t) = \boldsymbol{C}(r) + (t - t_0)\boldsymbol{D}(r) \tag{1}$$

where $\boldsymbol{C}(r)$ is called directrix curve and $\boldsymbol{D}(r)$ is a vector field along $\boldsymbol{C}(r)$. For fixed values r, this parameterization represents the straight line L on P.

The parameterization P describes exactly a developable surface if the condition $\det(\boldsymbol{C}', \boldsymbol{D}, \boldsymbol{D}') = 0$ is identically satisfied, where $\boldsymbol{C}'$ and $\boldsymbol{D}'$ denote the derivatives of $\boldsymbol{C}(r)$ and $\boldsymbol{D}(r)$, respectively. There are three types of developable surfaces: cones, cylinders (including planes), and tangent surfaces formed by the tangents of a space curve, which is called the cuspidal edge, or the edge of regression. Cylinders do not contain singular points. The only singular point of a cone is its vertex. The singular points of a tangent surface coincide with its cuspidal edge.

3. Parametric representation of developable surfaces passing through a given curve

3.1. The case for passing through an arbitrary curve

Given a spatial parametric curve

$$R: \boldsymbol{R}(r) = (X(r), Y(r), Z(r)), \quad 0 \leqslant r \leqslant H$$

Without losing generality, we assume that it has third derivatives, and $\boldsymbol{R}'(r) \times \boldsymbol{R}''(r) \neq 0$, because otherwise the curve is a straight line segment or the principal normal is undefined at some points on the curve. Then the components of its Frenet frame are defined as [11]

$$\boldsymbol{T}(r) = \frac{\boldsymbol{R}'(r)}{|\boldsymbol{R}'(r)|}$$
$$\boldsymbol{B}(r) = \frac{\boldsymbol{R}'(r) \times \boldsymbol{R}''(r)}{|\boldsymbol{R}'(r) \times \boldsymbol{R}''(r)|}$$
$$\boldsymbol{N}(r) = \boldsymbol{B}(r) \times \boldsymbol{T}(r)$$

The isoparametric surfaces generated from the arbitrarily parameterized curve $\boldsymbol{R}(r)$ are expressed as [10]

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (U(r,t), V(r,t), W(r,t))(\boldsymbol{T}(r), \boldsymbol{N}(r), \boldsymbol{B}(r))^{\mathrm{T}}, \quad 0 \leqslant r \leqslant H, \ 0 \leqslant t \leqslant T \tag{2}$$

where $U(r,t)$, $V(r,t)$, and $W(r,t)$ are all C^1 functions, called the marching-scale functions in the directions $\boldsymbol{T}(r)$, $\boldsymbol{N}(r)$, and $\boldsymbol{B}(r)$, respectively. The values of $U(r,t)$, $V(r,t)$, and $W(r,t)$ indicate the extension-like, flexion-like, and torsion-like effects, respectively, by the point unit through the time t, starting from $\boldsymbol{R}(r)$.

Our aim is to find the necessary and sufficient conditions for the surface represented by Eq. (2) to be developable. First, since $\boldsymbol{R}(r)$ is an isoparametric curve on the surface $\boldsymbol{P}(r,t)$, this is equivalent to the case that there exists a parameter $t_0 \in [0, T]$ such that $\boldsymbol{P}(r, t_0) = \boldsymbol{R}(r)$, $0 \leqslant r \leqslant H$.

Secondly, suppose $\boldsymbol{P}(r,t)$ is a ruled surface represented by Eq. (1). Substituting $t = t_0$ into Eq. (2), immediately we have that $\boldsymbol{R}(r)$ must be the directrix, and

$$(t - t_0)\boldsymbol{D}(r) = (U(r,t), V(r,t), W(r,t))(\boldsymbol{T}(r), \boldsymbol{N}(r), \boldsymbol{B}(r))^{\mathrm{T}}$$

The above equation indicates that there exist three functions which are only dependent on the parameter r: $u(r)$, $v(r)$, and $w(r)$, so that $U(r,t)$, $V(r,t)$, and $W(r,t)$ can be decomposed into the multiplication of $t - t_0$ and $u(r)$, $v(r)$, and $w(r)$, respectively:

$$(U(r,t), V(r,t), W(r,t)) = (t - t_0)(u(r), v(r), w(r))$$

Then Eq. (2) is rewritten as

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (t - t_0) \times (u(r), v(r), w(r)) \cdot (\boldsymbol{T}(r), \boldsymbol{N}(r), \boldsymbol{B}(r))^{\mathrm{T}} \tag{3}$$

Thirdly, $\boldsymbol{P}(r,t)$ is developable if and only if there is $\det(\boldsymbol{R}', \boldsymbol{D}, \boldsymbol{D}') = 0$. A simple computation shows that, for a regular curve $\boldsymbol{R}(r)$, the equation is equivalent to

$$(vw' - wv') - \kappa uw + \tau(v^2 + w^2) = 0 \tag{4}$$

where κ and τ represent the curvature and the torsion of $\boldsymbol{R}(r)$, respectively, and u, v, w are shortening for functions $u(r)$, $v(r)$, $w(r)$, while u', v', w' for their derivatives, respectively. When $w \neq 0$, the above equation can be rewritten as

$$\left(\frac{v}{w}\right)' - \tau\left[\left(\frac{v}{w}\right)^2 + 1\right] + \kappa\left(\frac{u}{w}\right) = 0$$

Having specified $v(r)$ and $w(r)$, users can solve the expression of $u(r)$ conveniently using the above equation.

Then the following theorem can be derived.

Theorem 1. *The necessary and sufficient conditions for the surface $P(r,t)$ to be a developable surface passing through the* *given spatial isoparametric curve $R(r)$ are that there exist a parameter $t_0 \in [0, T]$ and the functions $u(r)$, $v(r)$, and $w(r)$, such that $P(r,t)$ can be expressed by Eq. (3), satisfying the condition shown in Eq. (4).*

Note that the above theorem still holds when substituting function $f(t)$ for the parameter t. To obtain a developable surface, we can first design the marching-scale functions in Eq. (4), and then apply them to Eq. (3) to derive the final parameterization. In what follows, we will provide the criteria to classify the developable surface according to the various forms of $u(r)$, $v(r)$, and $w(r)$.

For cylinders, the vector field is parallel to a fixed direction. Assume the vector field is $\bar{\boldsymbol{D}}$. From Eq. (3), we can decompose $\bar{\boldsymbol{D}}$ along the three orthogonal vectors $\boldsymbol{T}(r)$, $\boldsymbol{N}(r)$, and $\boldsymbol{B}(r)$, and their projections are $u(r)$, $v(r)$, and $w(r)$, respectively. Thus, the surface represented by Eq. (3) is a cylinder if and only if there exists a constant vector $\bar{\boldsymbol{D}}$, so that

$$u(r) : v(r) : w(r) = (\bar{\boldsymbol{D}} \cdot \boldsymbol{T}) : (\bar{\boldsymbol{D}} \cdot \boldsymbol{N}) : (\bar{\boldsymbol{D}} \cdot \boldsymbol{B})$$

For cones, there exists a fixed point, the vertex, which is the common intersection of all the generating lines. Suppose the point is A, the ruling lines are all parallel to the direction of the vector $\boldsymbol{A} - \boldsymbol{R}(r)$. Similarly, the surface represented by Eq. (3) is a cone if and only if there exists a fixed point A, so that

$$u(r) : v(r) : w(r) = [(\boldsymbol{A} - \boldsymbol{R}(r)) \cdot \boldsymbol{T}] : [(\boldsymbol{A} - \boldsymbol{R}(r)) \cdot \boldsymbol{N}] : [(\boldsymbol{A} - \boldsymbol{R}(r)) \cdot \boldsymbol{B}]$$

Assume that the ratio of the above equation is $1/g(r)$, that is, $u(r)/[(\boldsymbol{A} - \boldsymbol{R}(r)) \cdot \boldsymbol{T}] = 1/g(r)$. With proper ratio function, we can keep the singularity out of the designed surface. Since $\boldsymbol{P}(r, t_0 + g(r)) = \boldsymbol{A}$, that is, the vertex $\boldsymbol{A}$ corresponds to the parameter sequence $(r, t_0 + g(r))$, we only need to guarantee that $t_0 + g(r) \notin [0, T]$ when $r \in [0, H]$.

Tangent surface is composed of tangents of a spatial curve, the cuspidal edge, represented as $\boldsymbol{E}(r)$ in this study. Every ruling line marches along the direction of the vector $\boldsymbol{E}'(r)$. So the surface represented by Eq. (3) is a tangent surface if and only if there exists a curve $\boldsymbol{E}(r)$ so that

$$u(r) : v(r) : w(r) = (\boldsymbol{E}'(r) \cdot \boldsymbol{T}) : (\boldsymbol{E}'(r) \cdot \boldsymbol{N}) : (\boldsymbol{E}'(r) \cdot \boldsymbol{B})$$

Assume the ratio of the above equation is $1/g(r)$, that is, $u(r)/[\boldsymbol{E}'(r) \cdot \boldsymbol{T}] = 1/g(r)$. Since

$$\boldsymbol{E}(r) = \boldsymbol{P}\left(r, t_0 + \frac{(\boldsymbol{E}(r) - \boldsymbol{R}(r)) \cdot \boldsymbol{E}'(r)}{\boldsymbol{E}'(r) \cdot \boldsymbol{E}'(r)} g(r)\right)$$

we just need to select proper $g(r)$ to keep singularities out of the surface.

Besides the method mentioned above, the type of developable surface can be identified in a different way. The ruled surface represented by Eq. (3) is a cylinder if and only if $\boldsymbol{D}(r) \times \boldsymbol{D}'(r) = 0$, that is

$$\begin{aligned} &vw' - wv' - \kappa uw + \tau(v^2 + w^2) = 0 \\ &u'w - uw' - \tau uv - \kappa vw = 0 \\ &uv' - u'v - \tau uw + \kappa(u^2 + v^2) = 0 \end{aligned}$$

Known from Eq. (4), the solution of the above first equality constructs a developable surface. The solution satisfying all the three equalities constructs a cylinder or a cone when

$$\boldsymbol{E}(r) = \boldsymbol{R}(r) - \frac{\boldsymbol{R}'(r) \cdot \boldsymbol{D}'(r)}{\boldsymbol{D}'(r) \cdot \boldsymbol{D}'(r)} \boldsymbol{D}(r) \tag{5}$$

degenerates to a point. When $\boldsymbol{E}(r)$ represents a nondegenerate curve, the surface is a tangent surface.

All the three types of surfaces can be determined by solving Eq. (4) or its rewritten expression. Discriminating the solution functions $u(r)$, $v(r)$, and $w(r)$ by the above method, we can derive different types of surfaces with or without singularities on them.

3.2. The case for passing through a geodesic

As mentioned above, in shoemaking industry, the girth, which is used to measure the shoe size, is usually regarded as a geodesic on the shoe surface. And the material of shoe surface can be flattened to a plane with minimum or even no distortion, so it can be seen as developable. In this section, we will solve this problem: given a spatial curve, how to construct a developable surface with it as a geodesic? Wang et al. [10] provided the necessary and sufficient conditions for designing a surface through a given geodesic. Combining those conditions with developable requirements, we will derive the designing method.

Assuming the given curve is $\boldsymbol{R}(r)$, Wang et al. [10] introduced the necessary and sufficient conditions for a surface denoted by Eq. (2) with $\boldsymbol{R}(r)$ as an isogeodesic curve. Zhao et al. [12] simplified and rewrote them as follows: there exists a parameter $t_0 \in [0, T]$ satisfying

$$\begin{aligned} &u(r, t_0) = v(r, t_0) = w(r, t_0) = 0 \\ &\frac{\partial v(r, t_0)}{\partial t} = 0 \\ &\frac{\partial w(r, t_0)}{\partial t} \neq 0 \end{aligned}$$

Theorem 1 has given the necessary and sufficient conditions for constructing a developable surface through $\boldsymbol{R}(r)$. Together with the above conditions, we then have the following theorem:

Theorem 2. *The necessary and sufficient condition for $P(r,t)$ being a developable surface with $R(r)$ as an isogeodesic is that there exist a parameter $t_0 \in [0, T]$ and a function $g(r)$, so that $P(r,t)$ can be represented by*

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (t - t_0)g(r)(\tau\boldsymbol{T} + \kappa\boldsymbol{B}) \tag{6}$$

Theorem 2 shows that the property of the constructed developable surface is completely determined by the given geodesic, and so is the type of the surface. Since there are three types of developable surfaces, the given curve can be classified into three kinds correspondingly. In what follows, we will discuss the relationship between the given geodesic and its isoparametric developable surface.

Suppose the surface constructed by Eq. (6) is a cylinder, then there is $\boldsymbol{D}' \times \boldsymbol{D} = 0$, which results in $\kappa\tau' - \kappa'\tau = 0$, namely $(\kappa/\tau)' = 0$. The ratio of curvature to torsion is a constant, and so the curve is a generalized helix. Then the following conclusion could be drawn.

Corollary 3. *Planar curves or generalized helixes are isogeodesic to cylinders.*

Suppose the surface is a cone, then the expression in Eq. (5) degenerates to a constant vector, that is, $E'(r) = 0$. It results in $(\tau/\kappa)' = G\cdot\|R'\|$, where G is an arbitrary constant. Let $s(r)$ denote the arc length function of $R(r)$, then there is

$$\tau(r)/\kappa(r) - \tau(0)/\kappa(0) = G \cdot s(r) \tag{7}$$

Corollary 4. *Curves satisfying Eq. (7) are isogeodesic to cones.*

The curves not satisfying Corollaries 3 and 4 are isogeodesic to tangent surfaces.

Polynomials are preferred representation in engineering application, such as Bézier and B-spline surfaces. However, the developable surface represented by Eq. (6) is usually not polynomial due to the normalization of $\boldsymbol{T}(r)$ and $\boldsymbol{B}(r)$. The problem can be solved by selecting proper marching-scale functions. That is, we can choose $u(r) = g(r)\cdot\|\boldsymbol{R}'\|^3(\boldsymbol{R}', \boldsymbol{R}'', \boldsymbol{R}''')$, $w(r) = g(r)\cdot\|\boldsymbol{R}' \times \boldsymbol{R}''\|^3$, where $g(r)$ is an arbitrary-scale function specified by users. Then the surface is expressed as

$$\boldsymbol{P}(r,t) = \boldsymbol{R}(r) + (t - t_0)g(r)[(\boldsymbol{R}')^2(\boldsymbol{R}', \boldsymbol{R}'', \boldsymbol{R}'') \cdot \boldsymbol{R}' + (\boldsymbol{R}' \times \boldsymbol{R}'')^2 \cdot (\boldsymbol{R}' \times \boldsymbol{R}'')]$$

Particularly, $g(r)$ can be selected to be a constant. Then for a degree-n Bézier curve, the degree of its isogeodesic surface is $(6n - 12) \times 1$ generally. When $n = 3$, it is a 6×1 surface.

4. Examples

In this section, we will illustrate the method and verify the conclusions. We first will derive the exact developable surface pencil whose members include cylinders, cones and tangent surfaces from a given helix. And then we will discuss the case when the curve is given as a geodesic.

Given a circular helix represented by Fig. 1

$$\boldsymbol{R}(r) = (a\cos r, a\sin r, br), \quad a > 0,\ b \neq 0,\ 0 \leqslant r \leqslant 2\pi$$

the Frenet trihedron frame is

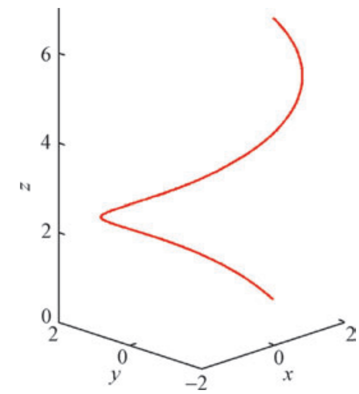

Fig. 1. The given helix with $a = 2$, $b = 1$.

$$\boldsymbol{T}(r) = (-a\sin r, a\cos r, b)/\sqrt{a^2 + b^2}$$
$$\boldsymbol{N}(r) = (-\cos r, -\sin r, 0)$$
$$\boldsymbol{B}(r) = (b\sin r, -b\cos r, a)/\sqrt{a^2 + b^2}$$

The parametric surface defined by Eq. (3) is

$$\begin{aligned}\boldsymbol{P}(r,t) = ((a - tv(r)))\cos r - \frac{t}{\sqrt{a^2+b^2}}(au(r) - bw(r))\sin r,\\ \times (a - tv(r))\sin r + \frac{t}{\sqrt{a^2+b^2}}(au(r) - bw(r))\cos r,\\ \times\ br + \frac{t}{\sqrt{a^2+b^2}}(bu(r) + aw(r))\end{aligned}$$

To design a developable surface, we have to solve Eq. (4). We can first specify $v(r)$ and $w(r)$, and then compute $u(r)$ from

$$u(r) = \frac{vw' - v'w + \tau(v^2 + w^2)}{\kappa w}$$

Given more information, such as the vector field direction, or the vertex, or the cuspidal edge, we can design a cylinder (Fig. 2), or a cone (Fig. 3), or a tangent surface (Fig. 4), respectively, according to the results in Section 3.1.

In what follows, we will discuss the construction of a developable surface with the given curve as a geodesic.

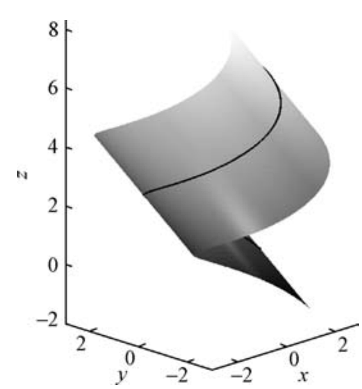

Fig. 2. Construction of a cylinder $\boldsymbol{P}(r,t)$ $(0 \leqslant r \leqslant 2\pi,\ 0 \leqslant t \leqslant 5)$ passing through the given helix.

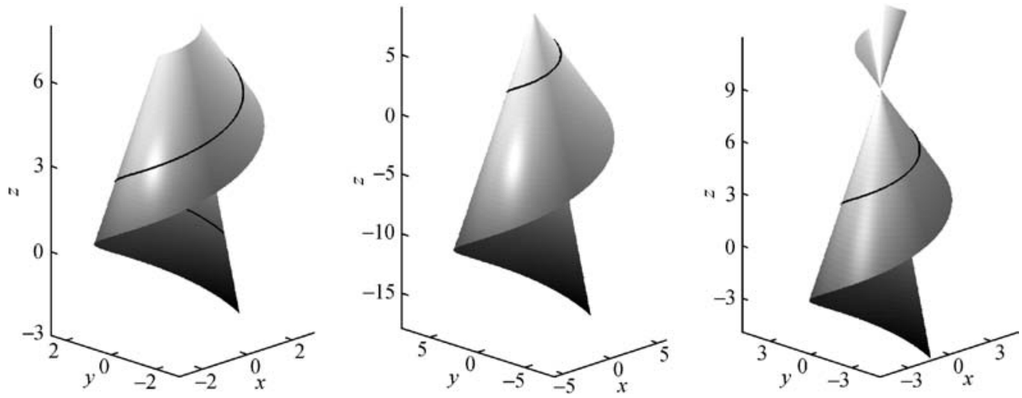

Fig. 3. Construction of cones $\boldsymbol{P}(r,t)$ $(0 \leqslant r \leqslant 2\pi,\ 0 \leqslant t \leqslant 5)$ passing through the given helix.

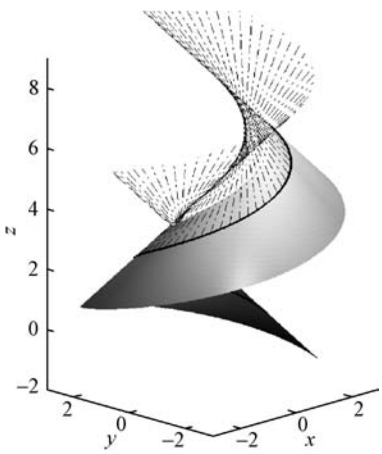

Fig. 4. Construction of a tangent surface $\boldsymbol{P}(r,t)$ $(0 \leqslant r \leqslant 2\pi,\ 0 \leqslant t \leqslant 5)$ passing through the given helix.

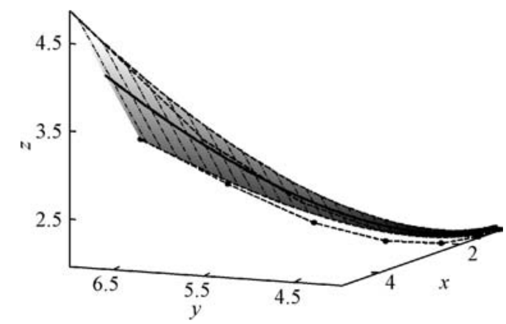

Fig. 6. A helix in cubic Bézier form is isogeodesic to a cylinder.

Corollary 3 states that a helix is isogeodesic to a cylinder. Wang et al. [10] gave an example of constructing an isogeodesic cylinder from a helix by choosing

$$u(r) = -\frac{b^2\sqrt{a^2+b^2}}{a^2}, \quad w(r) = -\frac{b\sqrt{a^2+b^2}}{a}$$

which in fact is a special case of Corollary 3, and so verifies the correctness of the results in this study.

Fig. 5 shows the construction of an isogeodesic cylinder from a planar cubic Bézier curve. It is a 6×1 Bézier surface. Besides, there is a kind of generalized helix which is isogeodesic to cylinders. They can be represented by polynomials. Suppose the expression is $\boldsymbol{R}(r) = \boldsymbol{A}_0 r^3 + \boldsymbol{A}_1 r^2 + \boldsymbol{A}_2\ r + \boldsymbol{A}_3$, where $\boldsymbol{A}_i (i = 0,1,2,3)$ are three coefficient vectors. When the three vectors $\boldsymbol{A}_i (i = 0,1,2)$ are perpendicular to each other, and $4\boldsymbol{A}_1^4 = 9\boldsymbol{A}_0^2\boldsymbol{A}_2^2$, the ratio of curvature to torsion is a constant, $\tau\ /\kappa = -3(\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2)/(2|\boldsymbol{A}_1|^3)$, so $\boldsymbol{R}(r)$ is a generalized helix. It is isogeodesic to a cylinder (Fig. 6).

Generally, when the given curve does not satisfy Corollaries 3 and 4, it is isogeodesic to a tangent surface (Fig. 7).

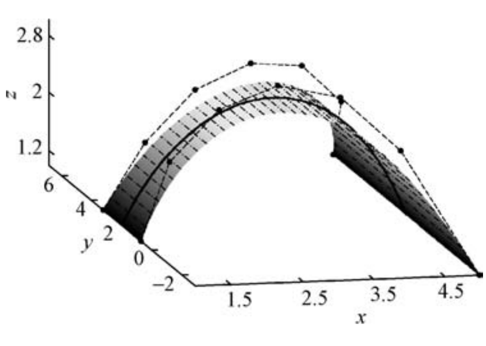

Fig. 5. Construction of a developable surface passing through a given cubic Bézier curve.

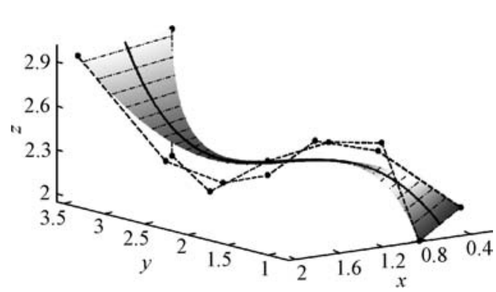

Fig. 7. The construction of a tangent surface with 6×1 degrees Bézier representation passing through the given cubic Bézier curve as an isogeodesic.

5. Conclusions

In this study, we have developed a method to design a developable surface using the surface pencil passing through a given curve. By representing the surface by the combination of the given curve, and the three vectors decomposed along the directions of Frenet trihedron frame, we derive the necessary and sufficient conditions for a surface to be developable. In addition, we have studied the problem of requiring the given curve to be a geodesic, which has potential applications in engineering. Research results show that the given curve can be classified into three kinds. They are, respectively, isogeodesic to different types of developable surface, that is, cylinders, cones and tangent surfaces. The theory and algorithm can be directly applied into CAD and CAM system which supports the shoemaking, garment-manufacture industry, and so on.

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